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A Closed Form Solution for the Proximity Effect in a Thin Tubular Conductor Influenced by a Parallel Filament

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Abstract: The present paper deals with the proximity effect in a system consisting of a thin tubular conductor and a filament. The integral equation for the current density in the tubular conductor is solved by assuming a solution in the form of an infinite Fourier series. By using this solution the a.c. to d.c. resistance ratio for the tubular conductor is also found in a closed form.

Keywords: Proximity effect, Tubular conductor, Filament, A.C. resistance.

1 Introduction

It is known that the current density of a time-varying current is not uniform over the cross section of a conductor – it increases toward the surface. This is known as the skin effect. If another conductor with a time-varying current is present, it will cause an additional change in the current density in the first conductor. This phenomenon is referred to as the proximity effect. If the currents have opposite directions, they will concentrate in the regions of the conductors facing each other. The proximity effect causes an additional increase of the conductor's resistance, i.e. additional power losses.

Analysis of the proximity effect is much more involved compared to the analysis of the skin effect, and there are very few cases where a closed form solution can be obtained. One of these is the system consisting of a thin tubular conductor and a filament with sinusoidal currents. The first solution in a closed form for this problem was found by Dwight [1]. He solved integral equation [2] for the current density in the tubular conductor by using the method of successive approximations. A numerical solution for this problem is obtained in [3], by assuming it in the form of a finite trigonometric series (finite Fourier series). The unknown coefficients in this series are found by the point-matching procedure. In this paper the integral equation for the current density in the tubular conductor is solved in a closed form by assuming the current density in the form of an infinite Fourier series with unknown coefficients. These

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coefficients are found by equating coefficients with trigonometric functions on both sides of the equation. The solution thus obtained coincides with Dwight's solution, but the procedure is much simpler.

2 Integral Equation for Current Density in Thin Tubular Conductor Influenced by Filament

The geometry of the problem is shown in Fig. 1.

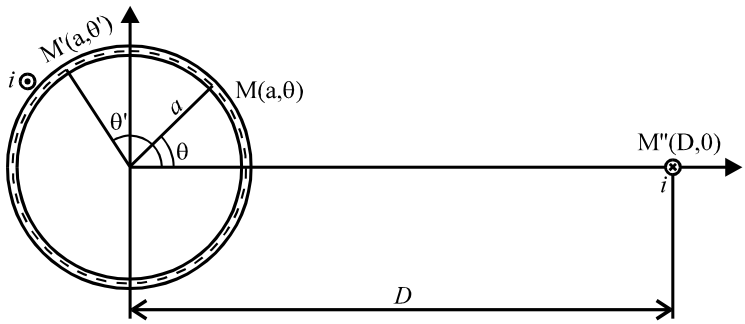

Fig. 1 – *Cross-section of a thin tubular conductor and a parallel filament.*

The radius of the thin tubular conductor is a, its thickness – d ($d << a$), and the distance between the conductor axis and the filament – D. Currents of angular frequency ω and r.m.s. value I flow through the tubular conductor and the filament in opposite directions. Since the tubular conductor is very thin, the radial dependence of its current density can be neglected, i.e. it is only a function of the polar angle θ.

The integral equation for the current density at an arbitrary point $M(r,\theta)$ of the tubular conductor is [3]

$$J(\theta) = \frac{\mathrm{j}\omega\mu_0\sigma}{4\pi}\left[\int_0^{2\pi} J(\theta')\ln\left(\overline{MM'}\right)^2 ad\,\mathrm{d}\theta' - I\ln\left(\overline{MM''}\right)^2\right] + K\,, \qquad (1)$$

where:

$$\left(\overline{MM'}\right)^2 = \left(a\cos\theta - a\cos\theta'\right)^2 + \left(a\sin\theta - a\sin\theta'\right)^2 = 4a^2\sin^2\frac{\theta-\theta'}{2}\,,$$

$$\left(\overline{MM''}\right)^2 = \left(a\cos\theta - D\right)^2 + a^2\sin^2\theta = a^2 + D^2 - 2aD\cos\theta\,,$$

and K is an unknown constant.

By dividing the quantities under both logarithms by D^2 and introducing

$$\lambda^2 = \pi f \mu_0 \sigma a d\ , \tag{2}$$

(1) becomes

$$J(\theta) = \frac{j\lambda^2}{2\pi}\left[\int_0^{2\pi} J(\theta')\ln\left(4\frac{a^2}{D^2}\sin^2\frac{\theta-\theta'}{2}\right)d\theta' - \right.$$
$$\left. -\frac{I}{ad}\ln\left(1+\left(\frac{a}{D}\right)^2 - \frac{2a}{D}\cos\theta\right)\right] + K. \tag{3}$$

3 Solution of Integral Equation for Current Density

By applying the superposition principle we seek for the solution of (3) in the form

$$J(\theta) = J_1(\theta) + J_2(\theta), \tag{4}$$

where $J_1(\theta)$ is the current density in the tubular conductor in the absence of the filament (skin effect), and $J_2(\theta)$ is the current density in the tubular conductor under the influence of the filament (proximity effect). In the first case we omit the middle term on the right-hand side of (3), and in the second case – constant K.

Since the thickness of the tubular conductor is very small, the skin effect is not present and therefore

$$J_1(\theta) = \frac{I}{S} = \frac{I}{2\pi a d} = \text{const}. \tag{5}$$

It remains to find $J_2(\theta)$ – a solution of (3) where constant K is omitted

$$J_2(\theta) = \frac{j\lambda^2}{2\pi}\left[\int_0^{2\pi} J_2(\theta')\ln\left(4\frac{a^2}{D^2}\sin^2\frac{\theta-\theta'}{2}\right)d\theta' - \right.$$
$$\left. -\frac{I}{ad}\ln\left(1+\left(\frac{a}{D}\right)^2 - \frac{2a}{D}\cos\theta\right)\right]. \tag{3'}$$

We seek a solution of (3') in the form of an infinite Fourier series

$$J_2(\theta) = \sum_{n=1}^{\infty} C_n \cos n\theta\,. \tag{6}$$

(Only cosine functions are taken, since the current density must be symmetrical with the respect to the x-axis).

By substituting (6) into (3'), and using the expansion (see Appendix 1)

$$\ln\left(1+\left(\frac{a}{D}\right)^2-\frac{2a}{D}\cos\theta\right)=-2\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^n\frac{\cos n\theta}{n} \tag{7}$$

we arrive at

$$\sum_{n=1}^{\infty} C_n \cos n\theta =$$

$$=\frac{\mathrm{j}\lambda^2}{2\pi}\left[\sum_{n=1}^{\infty} C_n \int_0^{2\pi}\cos n\theta' \ln\left(4\frac{a^2}{D^2}\sin^2\frac{\theta-\theta'}{2}\right)\mathrm{d}\,\theta' + \frac{2I}{ad}\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^n\frac{\cos n\theta}{n}\right].$$

The integral on the right-hand side can be evaluated in a closed form (see Appendix 2)

$$\int_0^{2\pi}\cos n\theta' \ln\left(4\frac{a^2}{D^2}\sin^2\frac{\theta-\theta'}{2}\right)\mathrm{d}\,\theta' = -2\pi\frac{\cos n\theta}{n},\ n\geq 1\,. \tag{8}$$

By substituting this result into the preceding formula and equating the coefficients with cos $n\theta$ we obtain

$$C_n = -\mathrm{j}\frac{\lambda^2}{2\pi}C_n\frac{2\pi}{n}+\mathrm{j}\frac{\lambda^2}{2\pi}\frac{2I}{ad}\left(\frac{a}{D}\right)^n\frac{1}{n},$$

whence

$$C_n=\frac{\mathrm{j}\dfrac{\lambda^2 I}{\pi ad}\left(\dfrac{a}{D}\right)^2}{n+\mathrm{j}\lambda^2}\,. \tag{9}$$

Finally, by making use of (5), (6) and (9), from (4) we obtain current density in the tubular conductor

$$J(\theta)=\frac{I}{2\pi ad}+\mathrm{j}\frac{\lambda^2 I}{\pi ad}\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^n\frac{\cos n\theta}{n+j\lambda^2}, \tag{10}$$

where λ^2 is given by (2). This is the same solution obtained by Dwight by a different way.

By using (10), we readily find the a.c. resistance of the tubular conductor per unit length.

$$R'_{a.c.} = \frac{P}{I^2} = \frac{1}{\sigma I^2}\int_V |J|^2 \, \mathrm{d}V = \frac{1}{\sigma I^2}\int_0^{2\pi} |J|^2 \, ad \cdot \mathrm{d}\theta =$$

$$= \frac{ad}{\sigma I^2}\int_0^{2\pi} J \cdot J^* \, \mathrm{d}\theta = \frac{1}{2\pi\sigma ad} + \frac{\lambda^4}{\pi\sigma ad}\sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^{2n}\frac{1}{n^2+\lambda^4},$$

or

$$\frac{R'_{a.c.}}{R'_{d.c.}} = 1 + 2\lambda^4 \sum_{n=1}^{\infty}\left(\frac{a}{D}\right)^{2n}\frac{1}{n^2+\lambda^4}. \tag{11}$$

Note: If the currents have the same direction, the second term on the right-hand side of (1) changes sign, and thus the second term in solution (10). Expression (11) for the a.c. to d.c. resistance ratio remains unchanged.

4 Numerical Results

Fig. 2 shows the normalized current distribution $\left|J(\theta)/(I/2\pi d)\right|$ versus the polar angle θ obtained from (10), for f = 50Hz, a = 5cm, d = 5mm, $\sigma = 57 \cdot 10^6$ S/m, for different values of the distance D between the conductors.

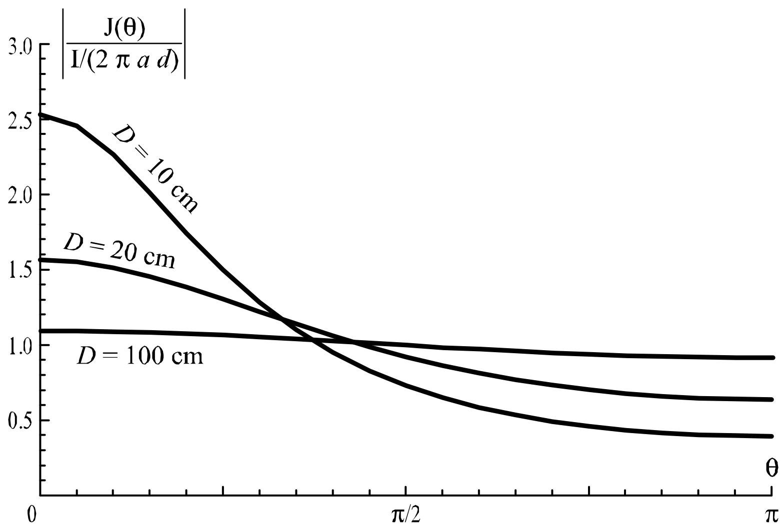

Fig. 2 – *Normalized current distribution in the tubular conductor for different values of distance D* (a = 5cm, d = 5mm, f = 50Hz).

If frequency f is taken as parameter, the corresponding plots of the normalized current density are shown in Fig. 3, for a = 5cm, d = 5mm and D = 10cm.

From (11) we have calculated the a.c. to d.c. resistance ratio versus frequency for different values of distance D. The corresponding plots are shown in Fig. 4, for a = 5cm and d = 5mm.

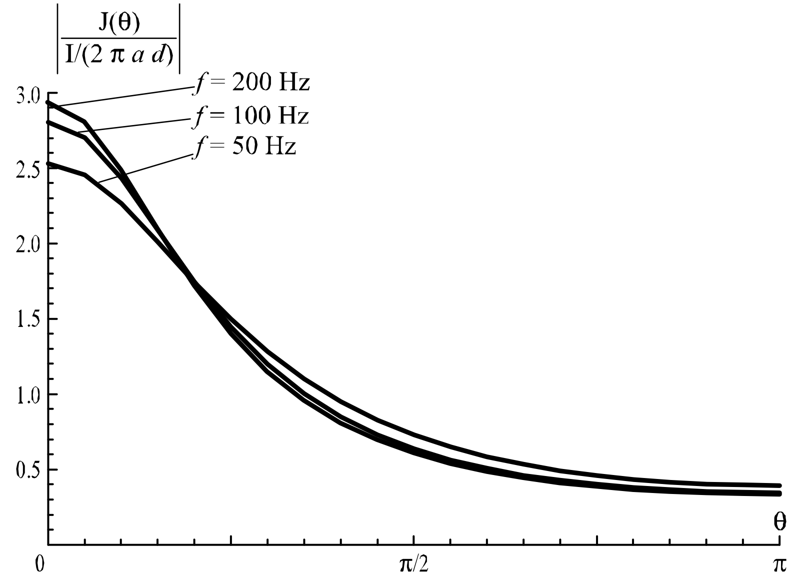

Fig. 3 – *Normalized current distribution in the tubular conductor for different values of frequency f* (a = 5cm, d = 5mm, D = 10cm).

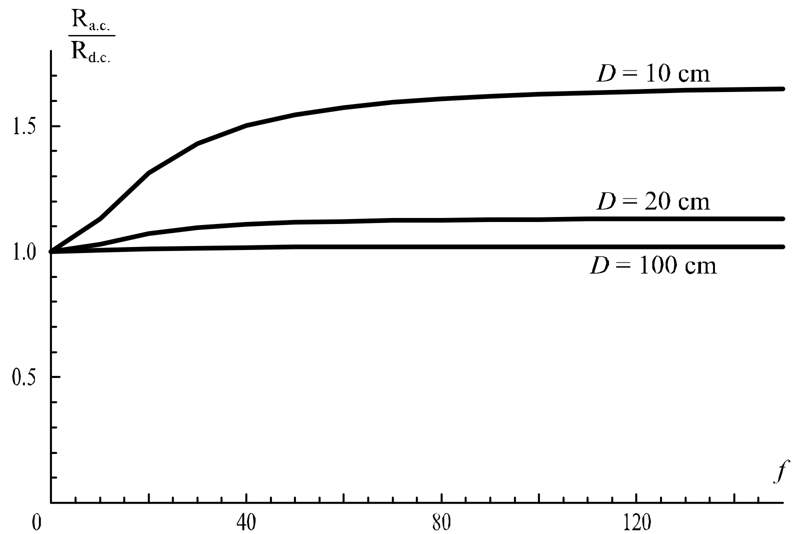

Fig. 4 – *a.c. to d.c. resistance ratio versus frequency f. Distance D is taken as a parameter* (a = 5cm, d = 5mm).

5 Conclusion

In this paper a closed-form solution for the proximity effect is found for the system consisting of a thin tubular conductor and a filament with equal sinusoidal currents of opposite directions. A solution for current density in the tubular conductor is assumed in the form of an infinite Fourier series with unknown coefficients. These coefficients are found by equating coefficients with trigonometric functions on both sides of the equation. By using the obtained solution, the a.c. to d.c. resistance ratio of the tubular conductor is also found in a closed form.

6 Appendix

6.1 Appendix 1

Here we prove expansion (7). The starting point is the well known complex Taylor's series

$$\ln\left(1-z\right)=-\sum_{n=1}^{\infty}\frac{z^n}{n},\ \left|z\right|<1\,. \tag{A1.1}$$

If we put $z=\dfrac{a}{D}\mathrm{e}^{j\theta}$, and equate real parts of the right- and left-hand sides of (A1.1), we arrive at (7).

6.2 Appendix 2

Here we prove (8). Let us denote:

$$I\left(\theta\right)=\int_0^{2\pi}\cos n\theta'\ln\left(\frac{4a^2}{D^2}\sin^2\frac{\theta-\theta'}{2}\right)\mathrm{d}\,\theta'\,. \tag{A2.1}$$

A partial integration gives:

$$I\left(\theta\right)=\frac{1}{n}\int_0^{2\pi}\sin n\theta'\cot\frac{\theta-\theta'}{2}\mathrm{d}\,\theta'\,. \tag{A2.2}$$

The integral in (A2.2) is evaluated in [4]:

$$\int_0^{2\pi}\sin n\theta'\cot\frac{\theta-\theta'}{2}\mathrm{d}\,\theta'=-2\pi\cos n\theta\,. \tag{A2.3}$$

Obviously, (A2.3), (A2.2) and (A2.1) imply (8).

7 References

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