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# Space Vector Representation of Induction Motor Model in Field Weakening Regime*

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**Abstract:** In this paper a space vector model of an induction motor in a field weakening regime is detailed. Stator and rotor flux space vector trajectories are shown in a complex plane for the case when the machine is supplied by limited voltage.

**Keywords:** Induction motor, Field Weakening, Space vector.

## 1 Introduction

Dynamic phenomena in the induction motor at voltage limit, i.e. when machine operates in field weakening, in literature is usually analyzed on the basis of linearized torque equation [1 – 5]. In this paper the general expression for torque changes during the single control period is obtained. Based on the proposed analysis, the trajectory of stator and rotor space vectors in field weakening in the complex plane is displayed. The proposed model is suitable for further synthesis of torque control structures in field weakening which are based on the voltage angle control.

## 2 Mathematical Model of Induction Motor

State space model of induction motor in stationary reference frame is given by following equations:

$$\underline{u}_{sk} = R_s \underline{i}_{sk} + \frac{\Delta \underline{\Psi}_{sk}}{\Delta T}, \tag{1}$$

$$\underline{u}_{rk} = R_r \underline{i}_{rk} + \frac{\Delta \underline{\Psi}_{rk}}{\Delta T} + \mathrm{j}\omega_{mk} \underline{\Psi}_{rk}, \tag{2}$$

$$\underline{\Psi}_{sk} = L_s \underline{i}_{sk} + M \underline{i}_{rk}, \tag{3}$$

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$$\underline{\Psi}_{rk} = L_r \underline{i}_{rk} + M \underline{i}_{sk}, \tag{4}$$

$$m_k = \frac{3}{2} P\left(\underline{\Psi}_{sk} \times \underline{i}_{sk}\right) = -\frac{3}{2} P \operatorname{Im}\left\{\underline{\Psi}_{sk} \underline{i}_{sk}^{*}\right\}, \tag{5}$$

where $\underline{u}_{sk}$, $\underline{i}_{sk}$, $\underline{\Psi}_{sk}$, $\underline{u}_{rk}$, $\underline{i}_{rk}$, $\underline{\Psi}_{rk}$ are stator and rotor voltages, currents and flux space vectors, respectively, $R_s$, $R_r$, $L_s$, $L_r$ are stator and rotor resistances and inductances, $M$ is mutual inductance, and $\omega_{mk}$ and $m_k$ are shaft speed and torque.

Changes in stator and rotor fluxes during the control period $kT$, $k = 0,1,2,...$ are:

$$\Delta\underline{\Psi}_{sk} = \underline{\Psi}_{s(k+1)} - \underline{\Psi}_{sk}, \tag{6}$$

$$\Delta\underline{\Psi}_{rk} = \underline{\Psi}_{r(k+1)} - \underline{\Psi}_{rk}. \tag{7}$$

Control variable at instant $k$, $k = 0,1,2,...$ is stator voltage space vector $\underline{u}_{sk}$. From (1) and (6) stator flux at next instant (with stator resistance neglected) is:

$$\underline{\Psi}_{s(k+1)} = \underline{\Psi}_{sk} + \Delta\underline{\Psi}_{sk} = \underline{\Psi}_{sk} + \underline{u}_{sk}\Delta T. \tag{8}$$

By using (1-8) stator flux space vectors at instants $k$ and $k+1$, ($\underline{\Psi}_{sk}$ and $\underline{\Psi}_{s(k+1)}$) can be plotted in stationary $\alpha-\beta$ reference frame, as shown in Fig.1.

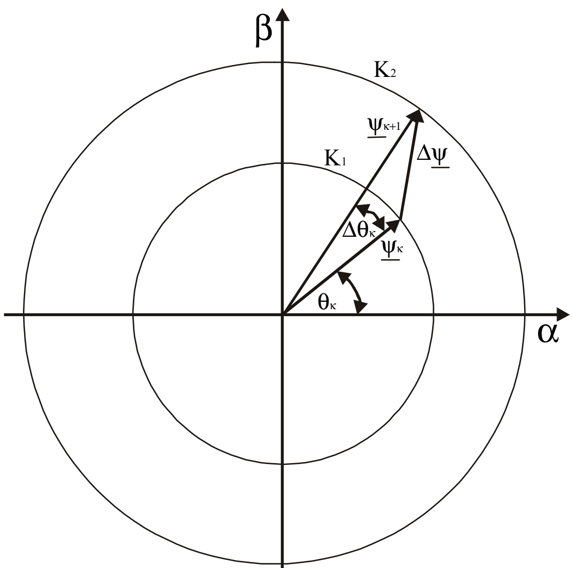


**Fig. 1 –** *Definition of stator space vectors.*

In general case, when $\left|\underline{\Psi}_{s(k+1)}\right| \neq \left|\underline{\Psi}_{sk}\right|$, tip of space vector $\underline{\Psi}_{s(k+1)}$ is on the circle $K_2$, which diameter can be different from diameter of circle $K_1$. In case $\left|\underline{\Psi}_{sk}\right| = \left|\underline{\Psi}_{s(k+1)}\right|$, tip of space vector $\underline{\Psi}_{s(k+1)}$ is on the circle $K_1$. The angle of stator flux space vector advancement during sample period $\Delta T$ is $\Delta\theta_k$ and in steady state it is equal to the product of synchronous speed and sample period:

$$\Delta\theta_k^0 = \omega_{sk}\Delta T\,. \tag{9}$$

Stator flux vector increment can be expressed by modulus and phase angle:

$$\Delta\underline{\Psi}_{sk} = \left|\underline{\Psi}_{s(k+1)}\right|\left(\cos\Delta\theta_k + \mathrm{j}\sin\Delta\theta_k\right) - \underline{\Psi}_{sk}\,, \tag{10}$$

and since the stator flux increment is proportional to the voltage (1):

$$\Delta\underline{\Psi}_{sk} \approx \underline{u}_{sk}\Delta T\,. \tag{11}$$

Stator flux vector increment and stator voltage can be expressed as:

$$\Delta\underline{\Psi}_{sk} = \left|\underline{u}_{sk}\right|\Delta T\left[\cos\left(\pi - \Delta\chi_k\right) + \mathrm{j}\sin\left(\pi - \Delta\chi_k\right)\right], \tag{12}$$

$$\underline{u}_{sk}^{\Psi_s} = \left|\underline{u}_{sk}\right|\left[\cos\left(\pi - \Delta\chi_k\right) + \mathrm{j}\sin\left(\pi - \Delta\chi_k\right)\right], \tag{13}$$

where $\Delta\chi_k$ is the angle between stator flux vector increment $\Delta\underline{\Psi}_{sk}$ and stator flux vector $\underline{\Psi}_{sk}$. The angles $\Delta\theta_k$ and $\Delta\chi_k$, as well as stator voltage $\underline{u}_{sk}^{\Psi_s}$ are referred to the stator flux space vector $\underline{\Psi}_{sk}$ as shown in Fig. 2.

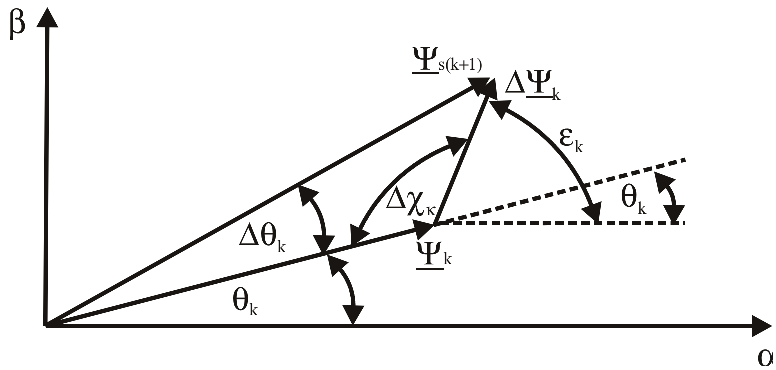


**Fig. 2 –** *Definition of stator flux space vector phase angles.*

Stator voltage (13), $\underline{u}_{sk}^{\Psi_s}$, is the control variable, and it is given by its amplitude $\left|\underline{u}_{sk}\right|$ and phase angle $\Delta\chi_k$ referred to the stator flux vector $\underline{\Psi}_{sk}$. In stationary reference frame, stator voltage is:

$$\underline{u}_{sk} = \left|\underline{u}_{sk}\right| \left[\cos\left(\theta_k + \pi - \Delta\chi_k\right) + \mathrm{j}\sin\left(\theta_k + \pi - \Delta\chi_k\right)\right]. \tag{14}$$

In base speed region, stator voltage is lower than rated, i.e. $\left|\underline{u}_{sk}\right| < U_{MAX}$, while in the field weakening regime stator voltage amplitude is constant and equal to the maximum available voltage, $\left|\underline{u}_{sk}\right| = U_{MAX}$. In the field weakening regime, the control variable is one of the angles of the triangle in Fig. 2, which is defined by space vectors $\Delta\underline{\Psi}_{sk}, \underline{\Psi}_{sk+1}, \underline{\Psi}_{sk}$.

Input variables to the model (1-5) are stator and rotor flux space vectors, $\underline{\Psi}_{sk}$ and $\underline{\Psi}_{rk}$, as well as shaft speed $\omega_{mk}$. Since the electrical transients are much faster than mechanical, it can be assumed that shaft speed is constant during the sample period, i.e. $\omega_{mk} = \omega_m = \mathrm{const.}$

## 3 Torque Changes During the Sample Period

Torque change during the sample period $k$, $k = 0,1,2,\dots$, is:

$$\Delta m_k = m_{k+1} - m_k, \tag{15}$$

and can be calculated from the model (1-5) if the initial state (stator and rotor fluxes and speed) and voltage command are given. However, the resulting expression would have no practical value. Therefore, simplified approaches based on the linearization of the torque equation (5) are used in the literature. Those expressions from [2 – 5] are incomplete since some parts of linearized equation are often neglected, or the shaft speed is used as parameter.

The expression for torque change during the sample period can be calculated from the definition (15) as:

$$\Delta m_k = \frac{3}{2} P \frac{L_m}{\sigma L_s L_r} \left[\left(\underline{\Psi}_{r(k+1)} \times \underline{\Psi}_{s(k+1)}\right) - \left(\underline{\Psi}_{rk} \times \underline{\Psi}_{sk}\right)\right]. \tag{16}$$

By eliminating stator currents from the model (1-5) and using space vector angles from Fig. 2. Stator current change during the sample period is:

$$\Delta\underline{i}_{sk} = \frac{\underline{u}_{sk} - \underline{e}_k}{\sigma L_s} \Delta T, \tag{17}$$

where back emf and slip $\omega_{klk}$ are:

$$\underline{e}_k = \frac{L_m}{L_r} j\omega_{sk} \underline{\Psi}_{rk} = \mathrm{j}\omega_{sk}\left(\underline{\Psi}_{sk} - \sigma L_s \underline{i}_{sk}\right). \tag{18}$$

$$\omega_{klk} = \omega_{sk} - \omega_{mk}. \tag{19}$$

Torque change during as function the voltage phase angle $\Delta\chi_k$ during the sample period is found from (16-19) as:

$$\Delta m_k = \frac{3P(1-\sigma)\left|\underline{\Psi}_{sk}\right|\Delta T}{2\sigma L_s\left[1+\sigma^2\omega_{klk}^2 T_r^2\right]}\Big[\left|\underline{u}_{sk}\right|\left(1+\sigma T_r\omega_{klk}\omega_{sk}\Delta T\right)\sin\Delta\chi_k - \\ -\left|\underline{u}_{sk}\right|\left(\omega_{sk}\Delta T-\sigma T_r\omega_{klk}\right)\cos\Delta\chi_k - \left|\underline{\Psi}_{sk}\right|\omega_{sk}\Big]. \tag{20}$$

The solutions of (20) are:

$$\Delta\chi_{k1} = -\arctan\frac{\omega_{sk}\Delta T-\sigma T_r\omega_{klk}}{1+\sigma T_r\omega_{klk}\omega_{sk}\Delta T} + \\ +\arcsin\frac{\dfrac{2\sigma L_s\left[1+\omega_{klk}^2\sigma^2 T_r^2\right]}{3P(1-\sigma)\left|\underline{\Psi}_k\right|\Delta T}\Delta m_k + \left|\underline{\Psi}_k\right|\omega_{sk}}{\left|\underline{u}_{sk}\right|\sqrt{\left(1+\sigma T_r\omega_{klk}\omega_{sk}\Delta T\right)^2+\left(\omega_{sk}\Delta T-\sigma T_r\omega_{klk}\right)^2}}, \tag{21}$$

$$\Delta\chi_{k2} = \pi-\arctan\frac{\omega_{sk}\Delta T-\sigma T_r\omega_{klk}}{1+\sigma T_r\omega_{klk}\omega_{sk}\Delta T} - \\ -\arcsin\frac{\dfrac{2\sigma L_s\left[1+\omega_{klk}^2\sigma^2 T_r^2\right]}{3P(1-\sigma)\left|\underline{\Psi}_k\right|\Delta T}\Delta m_k + \left|\underline{\Psi}_k\right|\omega_{sk}}{\left|\underline{u}_{sk}\right|\sqrt{\left(1+\sigma T_r\omega_{klk}\omega_{sk}\Delta T\right)^2+\left(\omega_{sk}\Delta T-\sigma T_r\omega_{klk}\right)^2}}. \tag{22}$$

It is clear from (21) and (22) that the same torque can be obtained by two different phase angles. Maximum change in torque increment (20) is found from:

$$\frac{\partial\left(\Delta m_k\right)}{\partial\left(\Delta\chi_k\right)} = 0, \tag{23}$$

For the voltage angle:

$$\Delta\chi_{kMAX} = \arctan\frac{\left(1+\sigma T_r\omega_{klk}\omega_{sk}\Delta T\right)}{\left(\omega_{sk}\Delta T-\sigma T_r\omega_{klk}\right)}. \tag{24}$$

It is important to note that the mean value of zeros of (20) is equal to the angle of maximum torque (24). This means that the angle at which torque increment has the maximum change is on the bisector of the angle defined by (21) and (22).

## 4 Space Vector Trajectories in Complex Plane

In the preceding section the expression for the torque change as the function of stator voltage module and phase angle is derived. The obtained expression can be represented graphically in the complex $\alpha-\beta$ plane by using corresponding space vectors at two successive instants $k$ and $k+1$. Steady state in torque, stator flux and synchronous speed is adopted as reference state, in which there is no change in torque and flux modulus at instants $k$ and $k+1$. In the reference state, the stator fluxes will have the same amplitudes in the current and next instant. Their movement will be defined by the angle (9), which is defined by the current synchronous speed. When the control variable $\Delta\chi_k$ is changed, all the values are going to be changed from the steady state.

In accordance with the adopted model, the stator and rotor fluxes in the two successive time instants will move by the angle equal to the product of synchronous speed and sample time. In order to torque being not changed in the reference state, the (25) must be satisfied:

$$\left|\underline{\Psi}_{s(k+1)}\right| \sin\theta_{rs(k+1)} = \left|\underline{\Psi}_{sk}\right| \sin\theta_{rsk} = h\,. \tag{25}$$

The condition (25) is shown in Fig. 3, which is the height of the parallelogram defined by stator and rotor space vectors. This condition defines the constant torque line $\Delta m_k = 0$ shown in Fig. 3.

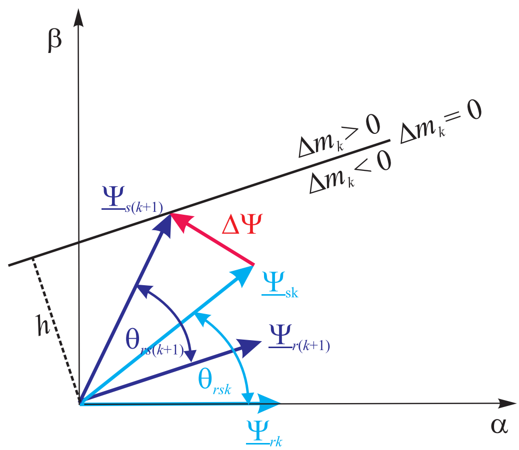


**Fig. 3 –** *The constant torque line.*

Constant torque line, i.e., $\Delta m_k = 0$ in complex $\alpha-\beta$ plane is parallel with rotor flux space vector at instant $k+1$ and passes through the tip of stator flux space vector. If the tip of stator flux space vector $\underline{\Psi}_{s(k+1)}$ at instant $k+1$ is on the constant torque line, the motor has the same torque at instants $k$ and $k+1$. If the tip of stator flux space vector is below the constant torque line, the torque at instant $k+1$ is lower, and if the tip is above the constant torque line, the torque at instant $k+1$ is larger than at instant $k$.

In reference state, stator and rotor flux space vectors move along circles with constant diameters, $\left|\underline{\Psi}_{s(k+1)}\right| = \left|\underline{\Psi}_{sk}\right|$ and $\left|\underline{\Psi}_{r(k+1)}\right| = \left|\underline{\Psi}_{rk}\right|$ with centres in origin. From Fig. 1, constant voltage amplitude is represented by circle with constant diameter $\left|\underline{u}_{sk}\right| = U_{MAX}$ with centre on the tip of stator flux space vector $\underline{\Psi}_{sk}$. Constant voltage and constant flux circles are shown in Fig. 4 together with constant torque line.

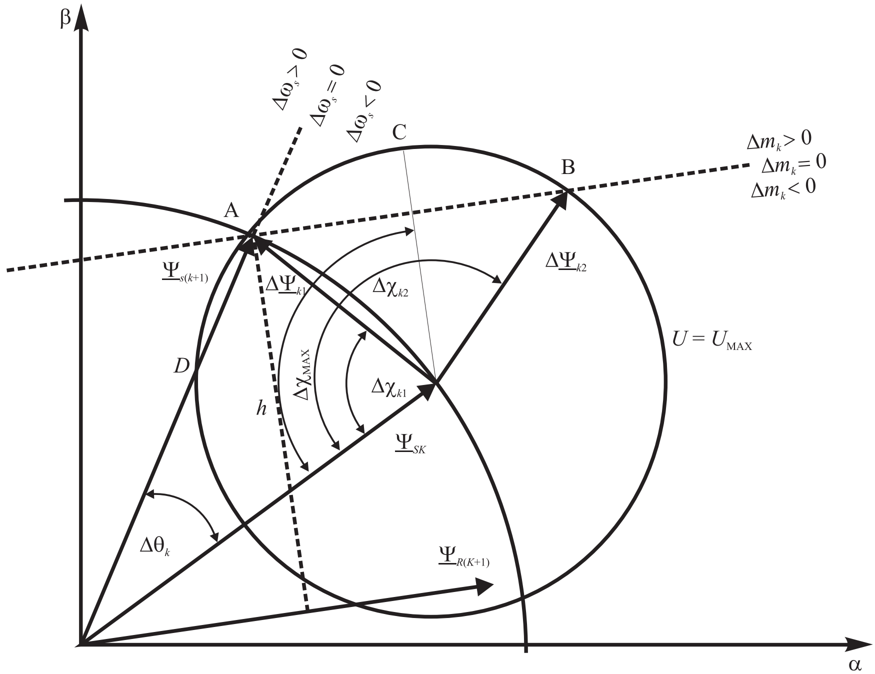


**Fig. 4 –** *Constant voltage, constant flux circles and constant torque and synchronous speed lines.*

Steady state (point A) is defined with control angle $\Delta\chi_{k1}$ (21), while point B is defined by angle $\Delta\chi_{k2}$ (22). Point C lies on the bisector of the angle between space vectors $\Delta\underline{\Psi}_1$ and $\Delta\underline{\Psi}_2$. Since the angle $\Delta\chi_k$ at point C is:

$$\Delta\chi_1 + \frac{\Delta\chi_2 - \Delta\chi_1}{2} = \frac{\Delta\chi_2 + \Delta\chi_1}{2} = \Delta\chi_{MAX} . \tag{26}$$

The maximum torque change is at point C in which the phase angle is $\Delta\chi_{MAX}$ (24). At point C the constant torque line is tangent line to the constant voltage circle.

The constant synchronous speed line $\Delta\omega_{sk} = 0$ is shown in Fig. 4 also. There are two points where the synchronous speed is at stationary value. The point A is steady state in all values, but at point D the torque is less than in the reference state. When the motor is unloaded at instant $k$, $k = 0,1,2,\ldots$, (when the slip is equal to zero, $\omega_{klk0} = 0$), maximum torque change is defined by angle:

$$\Delta\chi_{kMAX0} = \pi - \arctan\left(1/\omega_{sk}\Delta T\right) \approx \pi/2, \tag{27}$$

while when the machine is loaded with break-down torque (the slip is $\omega_{klkpr} = R_r / \sigma L_r$), the maximum torque change is defined by angle:

$$\Delta\chi_{kMAXpr} = \pi - \arctan\frac{1+\omega_{sk}\Delta T}{\omega_{sk}\Delta T - 1} \approx \frac{3\pi}{4}. \tag{28}$$

Complete trajectory of stator flux space vector in field weakening is shown in Fig. 5. At the beginning of the transient stator flux space vector is equal to $\underline{\Psi}_s(0)$ and at the end of transient it is equal to $\underline{\Psi}_s(N)$.

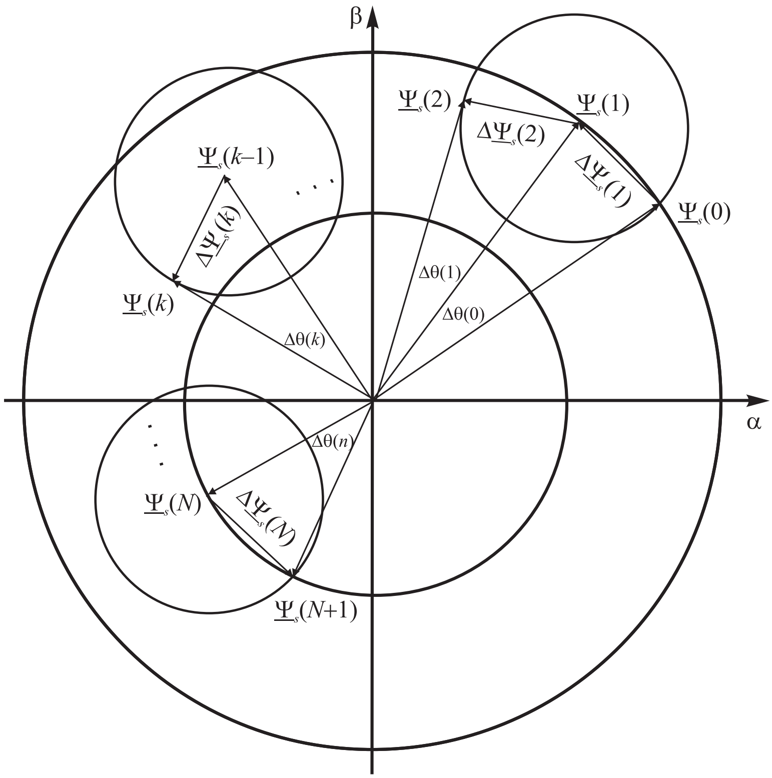


**Fig. 5 –** *Stator flux space vector trajectories in field weakening.*

During the transient, stator voltage has constant amplitude:

$$\left|\Delta\underline{\Psi}_s\left(k\right)\right| = U_{MAX}\Delta T = \text{const}, \quad k = 0,1,2,...,N \,. \tag{29}$$

Since the stator voltage is limited, the tips of the stator flux space vector increments lie on the circles defined by maximum available voltage. The only control variable is stator voltage phase angle. The tip of stator flux space vector moves along the spiral trajectory from the initial to the final position. The spiral motion of the stator flux space vector enables aperiodic response of torque and flux in field weakening [6].

## 5 Conclusion

In this paper the analysis of induction motor based on space vector representation is given. It is shown that, at voltage limit, the only independent control variable is voltage phase angle. The derived model is suitable for the synthesis of control procedure for induction motor control in the field weakening regime.

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