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Application of Hybrid Boundary Element Method – Example of Semishperical Ground Inhomogeneity

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Abstract: One new, so-called hybrid boundary element method (HBEM) is presented in this paper. It is a recently proposed numerical method for stationary and quasi-stationary EM field analysis. The method application is illustrated on the example of solving the problem of modelling hemispherical ground inhomogeneity influence on grounding system. The applied procedure also includes using of quasi-stationary image-theory. The obtained results are compared with those ones based on using the Green's function for the point source inside semi-spherical inhomogeneities as well as with the results obtained by applying COMSOL program package.

Keywords: Green's function, Ground inhomogeneity, Grounding systems, Hybrid Boundary Element Method, Resistance, Method of Moments, Quasi-stationary EM field.

1 Introduction

In this paper one recently proposed numerical method, so-called hybrid boundary element method (HBEM) [1 – 3] is presented. The HBEM is mostly based on the equivalent electrode method [4, 5] and the point-matching method [6]. It includes matching of the potential, as well as matching of the normal component of the boundary surface electric field. The method is also applicable to quasi-stationary analysis of grounding systems. There is no need for numerical integration and treating singular and nearly singular integrals.

The HBEM application is illustrated on the problem of analyzing influence of ground inhomogeneities modeled as hemispherically-shaped domains on grounding systems in their vicinity. There are various ground inhomogeneities which can be approximated by semi-conducting hemispherical domain, as pillar foundations, ponds and small lakes. This is the reason that many authors (including some of the co-authors of this paper) recently published papers dealing with this topic [7 – 13].

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A single wire electrode inside the hemispherical semi-conducting inhomogeneity, placed in homogeneous ground [13], is analyzed in this paper using HBEM. This model can be used as approximation of pillar grounding system, where foundation is approximated with hemisphere, while single wire electrode can be assumed as an equivalent of a wire armature cage system [11] (it is carried out using the complex function theory [14]).

The geometry parameters of the grounding rod and foundation as well as the values for the electrical parameters of the concrete are taken from the official publications and standards [15 – 18].

The obtained results are compared with those ones using the Green's function for the point source inside semi-spherical inhomogeneities as well as with the results obtained by applying COMSOL program package.

For constant leakage current, the results obtained using HBEM are compared with the ones obtained using procedure given in [4, 10 – 12] (based on using quasi-stationary Green's functions for the current source placed inside semi-conducting hemisphere) and those ones given in [7].

2 Theoretical Background

The pillar grounding system consisting of a wire ferro armature, Fig. 1, is observed. It is placed inside the hemispherical semi-conducting inhomogeneity of radii r_S and specific conductivity σ_2, surrounded by homogeneous ground having specific conductivity σ_1.

The wire armature cage is formed of N_C parallel conductors whose length is marked by h, having a circular cross-section of radius r_0, and placed on the circle of radius a_C, (upper part of Fig. 1). The corresponding cylindrical coordinates r and z have been introduced.

The system of vertical conductors can be replaced by one vertical conductor of length h and circular cross-section of radius [10],

$$a_\mathrm{e} = a_\mathrm{C} \sqrt[N_\mathrm{C}]{\frac{N_\mathrm{C}\, r_0}{a_\mathrm{C}}} \quad \text{[14, Appendix 1].} \tag{1}$$

The HBEM application to analysis of the grounding system from Fig. 1 is illustrated in Fig. 2. The boundary surface between the hemisphere and the surrounding ground, i.e. unknown total charges distribution on it, is modeled by rings of charges Q_n, $n = 1,...N$. The rings of radii a_n and the wires cross-section radii r_n, are placed parallel to the ground surface at the depths h_n, $n = 1,...N$. Based on procedure for HBEM application it is [3]

$$r_n = \left(\frac{4}{\pi}\right) r_\mathrm{s} \sin(\Delta\theta / 4), \quad \Delta\theta = \frac{0.5\pi}{N+1}, \quad n = 1, 2, ..., N\,. \tag{2}$$

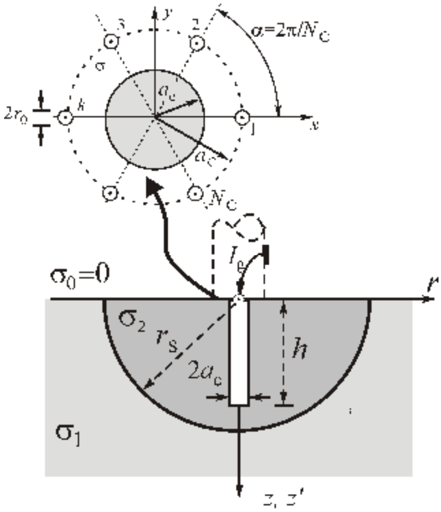

Fig. 1 – *Single electrode in the hemispherical domain.*

Application of HBEM includes uniform distribution of the rings, so that is, Fig. 2

$$a_n = r_\mathrm{s} \cos\theta_n, \quad h_n = r_\mathrm{s} \sin\theta_n, \quad \theta_n = n\Delta\theta = n\frac{0.5\pi}{N+1}, \quad n = 1, 2, ..., N\,. \tag{3}$$

The ring charges are

$$Q_n = \eta_n 2\pi r_n a_n \Delta\theta, \quad \Delta\theta = \frac{0.5\pi}{N+1}, \quad n = 1, 2, ..., N\,, \tag{4}$$

where η_n, $n = 1, 2, ..., N$, are total charges per unit surface at the corresponding surface domain.

The longitudinal current along the wire electrode is assumed in polynomial form as in [19]:

$$I(z') = \sum_{m=0}^{M} I_m \left(z'/h\right)^m. \tag{5}$$

while the leakage current is

$$I_\mathrm{leak}(z') = -\frac{\partial I(z')}{\partial z'} = -\sum_{m=1}^{M} \frac{mI_m}{l_k} \left(z'/h\right)^{m-1}. \tag{6}$$

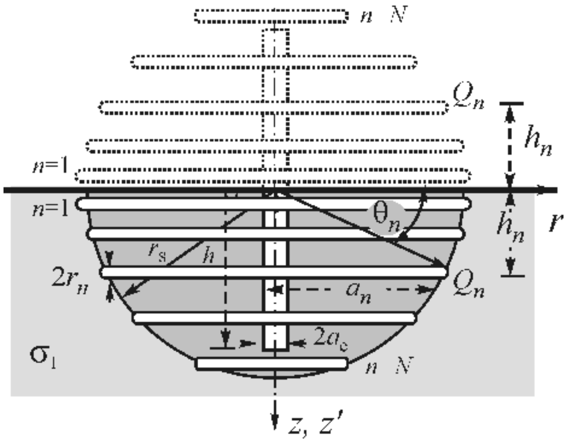

Fig. 2 – *Illustration for HBEM application.*

Applying the quasi-stationary image theory, the potential of the system versus cylindrical coordinates r. and z can be expressed as [3]

$$\phi(r,z)=\frac{1}{4\pi\sigma_2}\int_0^h I_{\text{leak}}(z')\left(\frac{1}{\sqrt{r^2+\left(z-z'\right)^2}}+\frac{1}{\sqrt{r^2+\left(z+z'\right)^2}}\right)\mathrm{d}z'+$$
$$\sum_{n=1}^{N}\frac{Q_n}{2\pi^2\varepsilon_0}\left(\frac{K\left(\pi/2,k_{1n}\right)}{\sqrt{\left(r+a_n\right)^2+\left(z-h_n\right)^2}}+\frac{K\left(\pi/2,k_{2n}\right)}{\sqrt{\left(r+a_n\right)^2+\left(z+h_n\right)^2}}\right), \tag{7}$$

where

$$K(\pi/2,k)=\int_0^{\pi/2}\left(1-k^2\sin^2\alpha\right)^{-1/2}\mathrm{d}\alpha \tag{8}$$

is a complete elliptic integral of the first kind, and corresponding modules k_{1n} and k_{2n} are defined as

$$k_{1n}^2=\frac{4r\,a_n}{\left(r+a_n\right)^2+\left(z-h_n\right)^2},\quad k_{2n}^2=\frac{4ra_n}{\left(r+a_n\right)^2+\left(z+h_n\right)^2}\,. \tag{9}$$

Now, there is a total of $M+1$ unknown current coefficients in (1) (I_m, $m=0,1,..,M$), and N charge coefficients in (3) (Q_n $n=1,2,..,N$), i.e. total of $M+N+1$ unknowns.

Based on the procedure from [3], $M+1$ equations can be formed satisfying the boundary condition for the potential given by (7) at the points on the wire electrode surface, i.e.

$$\phi(r_{k\mathrm{p}}, z_{k\mathrm{p}}) = U, \quad r_{k\mathrm{p}} = a_{\mathrm{e}}, \quad z_{k\mathrm{p}} = \frac{2k-1}{2(M+1)} h, \quad k = 1,2,...,M+1. \tag{10}$$

The rest of equations (N) is formed satisfying the boundary condition for the normal component of the electric field at the points placed on the hemispherical boundary surface between the two domains

$$\begin{gathered} E_{Rj}(R = r_s^+) = \frac{-\sigma_2}{\varepsilon_0(\sigma_1 - \sigma_2)} \eta_n, \quad E_{Rj} = E_{rj} \cos\theta j + E_{zj} \sin\theta j, \\ E_{rj} = -\frac{\partial\phi(r,z)}{\partial r}, \quad E_{zj} = -\frac{\partial\phi(r,z)}{\partial z}, \quad r = r_j, \quad z = z_j. \end{gathered} \tag{11}$$

In previous expression, R is radial coordinate of the spherical coordinate system, while r and z are coordinates of cylindrical coordinate system. The matching points are defined with the values of cylindrical coordinates

$$r_j = r_{\mathrm{s}} \cos\theta_j, \quad z_j = r_{\mathrm{s}} \sin\theta_j, \quad \theta_j = j \frac{0.5\pi}{N+1}, \quad j = 1,2,...,N. \tag{12}$$

After determining unknowns, it is possible to obtain the total feeding current as

$$I_g = \int_0^h I_{\mathrm{leak}}(z') \mathrm{d} z'. \tag{13}$$

Finally, the resistance of the grounding system can be determined as

$$R_g = \frac{U}{I_g}. \tag{14}$$

3 Numerical Results

Based on the presented model the corresponding program packages are developed and applied to approximate determining of the armature conductors' system placed inside the concrete foundation.

As a first step, the HBEM is applied on analysis of single wire electrode in the homogeneous ground ($\sigma_1 = \sigma_2$). The obtained results for assumed current distribution of the M-th degree given with (5) are compared in **Table 1** with those ones calculated using procedure based on segment current distribution (N_{segm} is segment number) [20] and the results obtained by COMSOL. The parameter values are $\sigma_1 = 0.01\ \mathrm{S/m}$, $r_{\mathrm{s}} = 2\ \mathrm{m}$, $h = 1\ \mathrm{m}$ and $a_{\mathrm{e}} = 3\ \mathrm{cm}$. One should notice that for assumed constant current distribution is $M = 1$, i.e. $N_{\mathrm{segm}} = 1$. The values for that case are identical, which was expected. Authors find interesting to remind that usually, for $M = N_{\mathrm{segm}}$, the procedure based on polynomial current distribution given with (5) gives more accurate results related to assumed segment current distribution. The number of the rings which are used for modelling semi-spherical surface is *N*=200. It is the value used for all calculations presented in this paper and it will not be further emphasized.

In order to further verify described procedure, the resistance of the electrode from Fig. 1 for parameter values: $\sigma_1 = 0.01\ \mathrm{S/m}$, $\sigma_2 = 0.05\ \mathrm{S/m}$, $r_\mathrm{s} = 1\ \mathrm{m}$, $h = 0.5\ \mathrm{m}$, and $a_\mathrm{e} = 1\ \mathrm{cm}$, is determined and compared with the results given in [7] (obtained using the procedure that considers application of the Boundary Elements Method) and [13] (obtained by applying the method from [10 – 14]). By that, leakage current distribution has been assumed as constant, i.e $M = 1$ in (5). Those results are shown in **Table 2**, which also includes the results obtained by COMSOL program package. The difference which exists between obtained results is acceptably small.

Table 1

Resistances values in homogeneous ground.

R_g [Ω]				
Polynomial approximation		Segments method		COMSOL
M			N_{segm}	
1	64.56	64.56	1	55.64
2	62.99	62.99	2	
3	62.24	62.24	3	
4	61.90	61.90	4	
5	61.63	61.63	5	
6	61.46	61.46	6	

Table 2

Comparison of the resistances values.

R_g [Ω]			
[7]	[13]	HBEM	COMSOL
40.2	41.15	36.82	36.3

The resistance values for $\sigma_1 = 0.01\mathrm{S/m}$, $\sigma_2 = 0.05\mathrm{S/m}$, $r_\mathrm{s} = 1\mathrm{m}$, $h = 0.5\mathrm{m}$, versus coefficient *M* in (5) are given in **Table 3**. The radii of the single wire takes values $a_\mathrm{e} = 1\mathrm{cm}$ [7] and $a_\mathrm{e} = 18\mathrm{cm}$ (the value determined using the procedure from Appendix and the parameters of the real grounding system from [1 – 3]).

Table 3

Resistances values versus coefficient M in (5).

R_g [Ω]				
M	a_e=1 cm		a_e=18 cm	
	HBEM	COMSOL		COMSOL
0	36.82	36.3	18.83	20.96
1	36.19		18.82	
2	35.89		17.80	
3	35.77		17.61	
4	35.66		17.47	
5	35.60		17.37	
6	35.54		17.28	

The electric scalar potential distribution, i.e. equipotential lines in the vicinity of the system from Fig. 1, normalized with the potential of the wire U are shown in Fig. 3. The parameter values are $\sigma_1 = 0.01$ S/m, $\sigma_2 = 0.05$ S/m, $r_s = 2\,\text{m}$, $h = 1\,\text{m}$ and $a_e = 3\,\text{cm}$. The normalized distribution of the rings total charges, which actually corresponds to the total charges per unit surface distribution at the semi-sphere surface for the same parameter values as in Fig. 3 can be seen in the Fig. 4. The normalization factor is $2\pi^2\varepsilon_0 hU$.

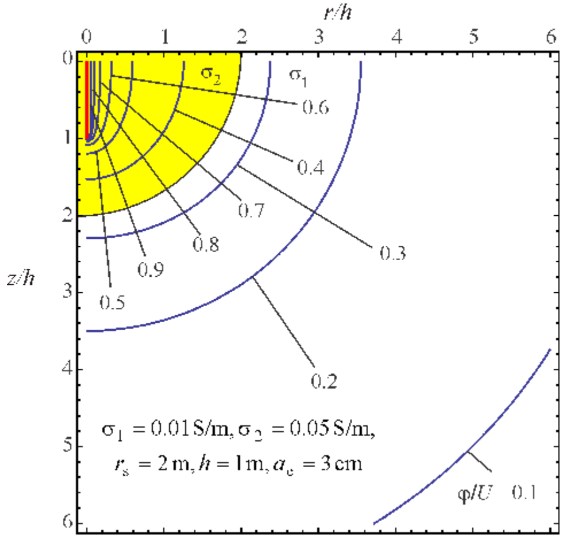

Fig. 3 – *Normalized electric scalar potential distribution in the vicinity of the system from* Fig. 1.

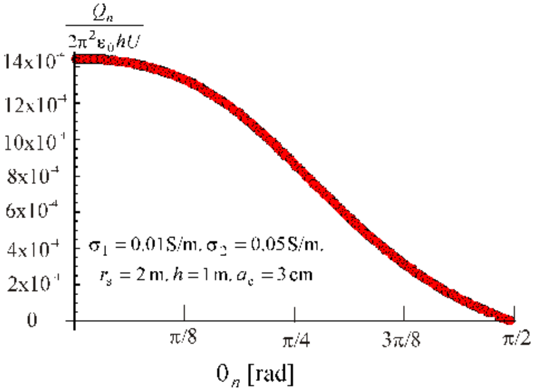

Fig. 4 – *Normalized total charge distribution at the surface of the semi-sphere.*

The behaviour of the functions given in Figs. 3 and 4 is expected. The potential value of the equipotential lines from Fig. 3 is increasing with approaching the single wire electrode from Fig. 1. Also, one can notice from Fig. 4 that the charge density value at the bottom point of the semi-sphere is zero, which is a logical result.

The resistance of the system from Fig. 1 versus ratio σ_1 / σ_2 is given in Fig. 5. Parameters' values are $\sigma_1 = 0.01\ \text{S/m}$, $\sigma_2 = 0.05\ \text{S/m}$, $r_s = 1\ \text{m}$, $h = 1\ \text{m}$ and $a_e = 1\ \text{cm}$.

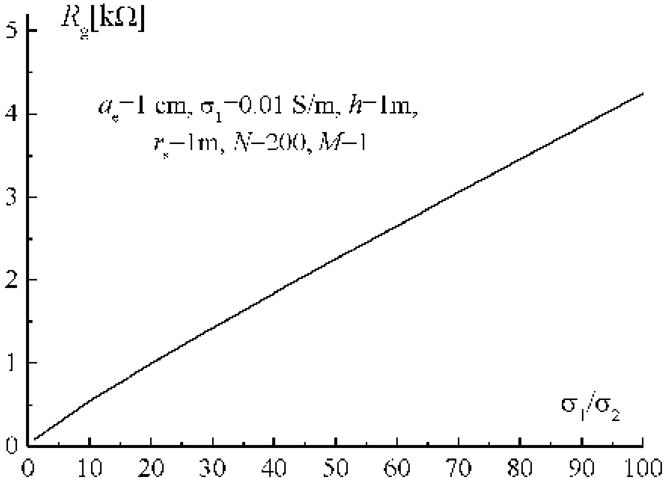

Fig. 5 – *Resistance of the system from* Fig. 1 *versus ratio* σ_1 / σ_2.

4 Conclusion

The procedure for applying one recently proposed numerical method, so-called hybrid boundary element method (HBEM) on grounding system analysis is presented in the paper. It is applied on characterization of the semi-spherically shaped ground inhomogeneity influence on the single wire grounding electrode placed inside of it. This model can be used as approximation of pillar foundations grounding system. In this case, single wire electrode is an electrical equivalent to a wire armature cage system and this approximation has been carried out using the complex function theory. The obtained results are compared with those ones obtained using other known methods and the solutions of the same problem existing in the literature. The accordance of the results is satisfactory. One can conclude that there is possibility to apply the HBEM for solving more complex ground inhomogeneity problems, which can not be solved comfortably, using existing ones.

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6 Appendix

In order to replace the system of vertical conductors having a circular cross-section of radius r_0 and placed on the circle of radius a_C, by a single wire vertical conductor of circular cross-section of radius a_e, Fig. A1, the complex function,

$$\underline{w} = C_1 \ln \underline{z} + C_2 \,, \ \underline{z} = r\,\mathrm{e}^{\mathrm{j}\theta} \tag{A1}$$

is used. It is applied on the analysis of the cage system formed from N conductors placed in linear homogeneous media of electrical conductivity σ and having leakage current of density per unit length I_leak,. Now, the complex potential is

$$\underline{w} = -\left(I_\mathrm{leak} / 2\pi\sigma\right) \sum_{k=1}^{N_\mathrm{C}} \ln(\underline{z} - \underline{z}_k) + C \,. \tag{A2}$$

In previous expression are $\underline{z}_k = a_{\mathrm{C}}\,\mathrm{e}^{\mathrm{j}(k-1)\alpha}$ and $\alpha = 2\pi/N_{\mathrm{C}}$, while C is constant depending on the referent level of zero potential. Since is

$$\underline{z}^{N_{\mathrm{C}}} - a^{N_{\mathrm{C}}} = \prod_{k=1}^{N_{\mathrm{C}}} (\underline{z} - \underline{z}_k) \tag{A3}$$

it obtains

$$\underline{w} = -\left(I_{\mathrm{leak}} / 2\pi\sigma\right)\ln(\underline{z}^{N_{\mathrm{C}}} - a^{N_{\mathrm{C}}}) + C. \tag{A4}$$

The electrical scalar potential is

$$\phi = \mathrm{Re}(\underline{w}) = -\frac{I_{\mathrm{leak}}}{4\pi\sigma N_{\mathrm{C}}}\ln(r^{2N_{\mathrm{C}}} + a_{\mathrm{C}}^{2N_{\mathrm{C}}} - 2\left(r a_{\mathrm{C}}\right)^{N_{\mathrm{C}}}\cos N_{\mathrm{C}}\theta) + C. \tag{A5}$$

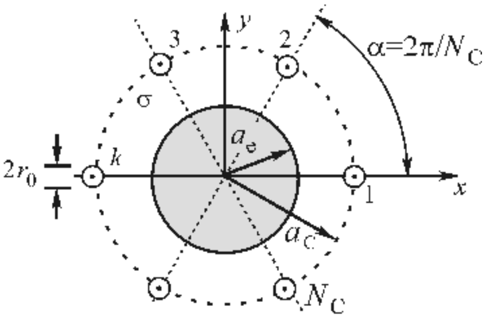

Fig. A1 – *Cage conductors' system.*

Applying previous expression for determining the potential of the single conductor at the point defined by $x = a_{\mathrm{C}} + r_0$, $y = 0$, i.e. $r = a_{\mathrm{C}} + r_0$, $\theta = 0$, obtains

$$U = \mathrm{Re}(\underline{w})\Big|_{\substack{x=a_{\mathrm{C}}+r_0 \\ y=0}} = -\frac{I_{\mathrm{tot}}}{2\pi\sigma N_{\mathrm{C}}}\ln((a_{\mathrm{C}} + r_0)^{N_{\mathrm{C}}} - R^{N_{\mathrm{C}}}) + C. \tag{A6}$$

In (A6), with $I_{\mathrm{tot}} = N_{\mathrm{C}} I_{\mathrm{leak}}$ the density per unit length of total leakage current from the cage system is labelled. Using condition $r_0 << 2a\pi/N_{\mathrm{C}}$ one can write

$$(a_{\mathrm{C}} + r_0)^{N_{\mathrm{C}}} \approx a_{\mathrm{C}}^{N_{\mathrm{C}}} + N_{\mathrm{C}} a_{\mathrm{C}}^{N_{\mathrm{C}}-1} r_0. \tag{A7}$$

and consequently, the potential of the single cage conductors' system is approximately

$$U = \mathrm{Re}(\underline{w})\Big|_{\substack{x=a_{\mathrm{C}}+r_0 \\ y=0}} = -\frac{I_{\mathrm{tot}}}{2\pi\sigma N_{\mathrm{C}}}\ln(N_{\mathrm{C}} a_{\mathrm{C}}^{N_{\mathrm{C}}-1} r_0) + C. \tag{A8}$$

If instead of the cage the single cylindrical electrode of cross-section having radii a_{e}, and leakage current of density per unit length I_{tot} is observed, the following expression can be formed,

$$U = -\frac{I_{\mathrm{tot}}}{2\pi\sigma N_{\mathrm{C}}}\ln(a_{\mathrm{e}}) + C\,. \tag{A9}$$

Comparing expressions (A8) and (A9) for the equivalent radius of the conductors' system from Fig. 1 obtains [10]

$$a_{\mathrm{e}} = a_{\mathrm{C}}\sqrt[N_{\mathrm{C}}]{\frac{N_{\mathrm{C}}\,r_0}{a_{\mathrm{C}}}}\,. \tag{10}$$

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