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Green's Function for the Semi-Infinite Strip in Terms of an Improper Integral

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Abstract: In this paper Green's function for the semi-infinite strip (which is the two-dimensional Green's function for a groove of infinite depth and length) is determined in the form of an improper integral, as opposed to the standard summation form. The integral itself, although rather complex, is found in a closed form. By using the derived Green's function simple formulas are obtained for a single and two-wire line configurations inside the groove.

Keywords: Semi-infinite strip, Groove, Green's function, Capacitance.

1 Introduction

Most commonly used methods for constructing Green's function for a given domain include separation of variables in Laplace's equation, method of images and method of conformal mappings (in the two-dimensional case) [1 – 3]. In this paper we use separation of variables in the two-dimensional Laplace's equation to determine Green's function for the semi-infinite strip, which is the cross-section of a groove of infinite depth and length. The sign of the separation constant is chosen so to give Green's function in the form of an improper integral contrary to the customary summation form. It turns out that the integral which determines Green's function can be evaluated in a closed form. By using the obtained Green's function, we provide some examples of capacitance calculations, including the capacitance of a single thin wire inside a grounded groove of infinite depth and length, and also the capacitances of some practical two-wire line configurations inside the groove. Special cases include the mentioned configurations above a ground plane or between two ground planes. Some of these formulas can be found in the handbook literature [4].

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2 Determination of Green's Function for the Semi-Infinite Strip

To derive Green's function for the semi-infinite strip of width a (Fig. 1) we have to find, by definition, the potential at the point (x, y) due to the unit line charge $q'=1$, passing through the point (x', y'), provided the boundary is kept at zero potential. We divide the strip into two domains by the plane $x=x'$, $0 \le y \le +\infty$, $-\infty \le z \le +\infty$, and we choose the sign of the separation constant so as to produce the harmonics sinhkx, coshkx, and the harmonics sinky, cosky. Then, the potentials in domains 1 and 2 can be sought in the form of integral combinations of these harmonics.

$$V_1(x,y)=\int_0^\infty A(k)\ \sinh kx\ \sin ky\,\mathrm{d}k, \quad 0\le x\le x', \tag{1}$$

$$V_2(x,y)=\int_0^\infty A(k)\ \frac{\sinh kx'}{\sinh k(a-x')}\ \sinh k(a-x)\sin ky\,\mathrm{d}k, \quad x'\le x\le a, \tag{2}$$

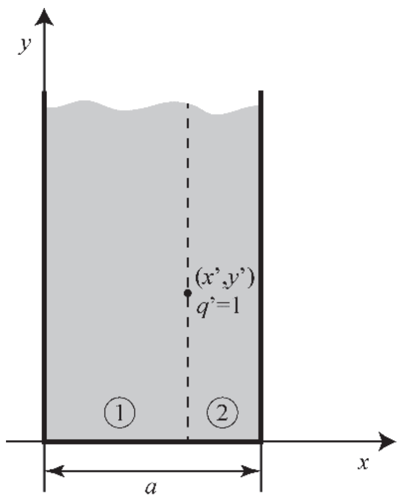

Fig. 1 – *Semi-infinite strip.*

where $A(k)$ is an unknown function. Clearly, the potentials given by (1) – (2) satisfy boundary conditions ($V_1 = 0$ for $x = 0$, $y = 0$ and $V_2 = 0$ for $x = a$, $y = 0$). Also, the potential is continuous across the interface ($V_1 = V_2$ for $x=x'$).

To find the unknown function $A(k)\,A(k)$ in (1) – (2) we use boundary condition for the normal components of the electric field at the interface $x=x'$:

$$\left.\left(\frac{\partial V_1}{\partial x}-\frac{\partial V_2}{\partial x}\right)\right|_{x=x'}=\frac{1}{\varepsilon_0}\delta(y-y'), \tag{3}$$

where the δ – function accounts for the presence of the unit line charge. Equation (3), after using (1) – (2), becomes

$$\int_0^{\infty} k\ A(k)\ \frac{\sinh ka}{\sinh k(a-x')}\ \sinh ky\ \mathrm{d}k = \frac{1}{\varepsilon_0}\delta(y-y')\,. \tag{4}$$

Next, we multiply (4) by $\sin my$ ($m > 0$), and integrate with respect to y from 0 to $+\infty$. This yields, after interchanging the order of integration

$$\int_0^{\infty} k\ A(k)\ \frac{\sinh ka}{\sinh k(a-x')}\mathrm{d}k\int_0^{\infty} \sin ky \sin my\ \mathrm{d}y = \frac{1}{\varepsilon_0}\sin my'\,. \tag{5}$$

The inner integral in (5) is evaluated as follows

$$\begin{aligned}\int_0^{\infty} \sin ky \sin my\ \mathrm{d}y &= \frac{1}{2}\int_0^{\infty} \cos(k-m)y\ \mathrm{d}y - \frac{1}{2}\int_0^{\infty} \cos(k+m)y\ \mathrm{d}y = \\ &= \frac{\pi}{2}\left[\delta(k-m)-\delta(k+m)\right].\end{aligned} \tag{6}$$

By substituting (6) into (5) we obtain the unknown function A

$$A(m) = \frac{2}{\pi\varepsilon_0}\ \frac{\sinh m(a-x')}{\sinh ma}\ \sin my'\,. \tag{7}$$

In deriving (5) – (7) we used the following properties of the δ function [5, 6]

$$\int_0^{\infty}\cos Ay\,\mathrm{d}y = \frac{1}{2}\int_{-\infty}^{\infty}\cos Ay\,\mathrm{d}y = \frac{1}{2}2\pi\delta(A) = \pi\delta(A)\,,$$

$$\int_0^{\infty} f(k)\delta(k-m)\,\mathrm{d}k = f(m), \quad m>0\,,$$

$$\int_0^{\infty} f(k)\delta(k+m)\,\mathrm{d}k = 0, \quad m>0\,.$$

Relation (7) determines potentials (1) – (2)

$$V_1(x,y) = V_1(x,x',y,y') = \frac{2}{\pi\varepsilon_0}\int_0^{\infty}\frac{\sinh k(a-x')}{k\sinh ka}\sin ky'\sinh kx\sin ky\,\mathrm{d}k\,, \tag{8}$$

$$\begin{aligned}V_2(x,y) = V_2(x,x',y,y') &= \frac{2}{\pi\varepsilon_0}\int_0^{\infty}\frac{\sinh kx'}{k\sinh ka}\sin ky'\sinh k(a-x)x\sin ky\,\mathrm{d}k = \\ &= V_1(x',x,y,y').\end{aligned} \tag{9}$$

Finally, Green's function is defined by

$$G(x,x',y,y')=\begin{cases}V_1(x,x',y,y'), & x\le x',\\ V_1(x',x,y,y'), & x\ge x'.\end{cases} \tag{10}$$

It remains to evaluate the integral in (8). This is done in the Appendix, resulting in

$$\int_0^\infty \frac{\sinh k(a-x')}{k\sinh ka}\sin ky'\sinh kx\sin ky\,\mathrm{d}k=$$
$$=\frac{1}{8}\ln\left[\frac{\cosh\frac{\pi}{a}(y-y')-\cos\frac{\pi}{a}(x+x')}{\cosh\frac{\pi}{a}(y-y')-\cos\frac{\pi}{a}(x-x')}\cdot\frac{\cosh\frac{\pi}{a}(y+y')-\cos\frac{\pi}{a}(x-x')}{\cosh\frac{\pi}{a}(y+y')-\cos\frac{\pi}{a}(x+x')}\right]. \tag{11}$$

Interchanging x and x' leaves (11) invariant, and Green's function (10), after using (8) and (11) gets its final form

$$G(x,x',y,y')=$$
$$=\frac{1}{4\pi\varepsilon_0}\ln\left[\frac{\cosh\frac{\pi}{a}(y-y')-\cos\frac{\pi}{a}(x+x')}{\cosh\frac{\pi}{a}(y-y')-\cos\frac{\pi}{a}(x-x')}\cdot\frac{\cosh\frac{\pi}{a}(y+y')-\cos\frac{\pi}{a}(x-x')}{\cosh\frac{\pi}{a}(y+y')-\cos\frac{\pi}{a}(x+x')}\right]. \tag{12}$$

In the standard procedure where the constant of separation is taken with the opposite sign, and the strip is divided by the plane $y = y'$, $0 \le x \le a$, $-\infty \le z \le +\infty$, then the proper harmonics are $\sin kx$, $\cos kx$ and $\sinh ky$, $\cosh ky$ (or $e^{\pm ky}$). In this case Green's function has the form of an infinite series [8], which, after summing, yields the same result, as given by (12).

3 Examples of Capacitance Calculations

In this section we use Green's function (12) to derive simple formulas for the capacitance of a single and two-wire line configurations inside a groove of infinite depth and length.

3.1 Capacitance of a single thin line conductor inside the grounded groove

A single thin line conductor inside a grounded groove is shown in Fig. 2.

Let the wire radius be R ($R \ll a$). We also assume that the conductor is not too close to the groove walls. The conductor potential is obtained from (12) if we put $x = x' - R = d - R$, $y = y' = H$ in the denominator of the first fraction under the logarithm sign; elsewhere we may take $x = x' = d$, $y = y' = H$.

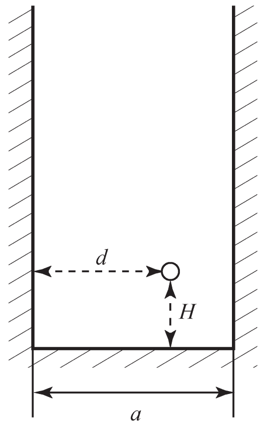

Fig. 2 – *Single line conductor inside the grounded groove of infinite depth and height.*

By using the approximation

$$\cosh\frac{\pi}{a}(y-y')-\cos\frac{\pi}{a}(x-x')=1-\cos\frac{\pi R}{a}\approx\frac{1}{2}\left(\frac{\pi R}{a}\right)^2,$$

we obtain

$$V_{cond}=\frac{1}{2\pi\varepsilon_0}\ln\left(\frac{2a}{\pi R}\frac{\sin\frac{\pi d}{a}\sinh\frac{\pi H}{a}}{\sqrt{\sin^2\frac{\pi d}{a}+\sinh^2\frac{\pi H}{a}}}\right), \tag{13}$$

and the capacitance is

$$C'=\frac{2\pi\varepsilon_0}{\ln\left(\frac{2a}{\pi R}\frac{\sin\frac{\pi d}{a}\sinh\frac{\pi H}{a}}{\sqrt{\sin^2\frac{\pi d}{a}+\sinh^2\frac{\pi H}{a}}}\right)}. \tag{14}$$

If we let $H\to+\infty$ in (14) we obtain the capacitance of a wire between two ground planes (Fig. 3).

$$C'=\frac{2\pi\varepsilon_0}{\ln\left(\frac{2a}{\pi R}\sin\frac{\pi d}{a}\right)}. \tag{15}$$

In principle, this special case can be treated by applying the method of images. However, the number of involved images is infinite, so that the procedure of finding the capacitance by images is not so simple.

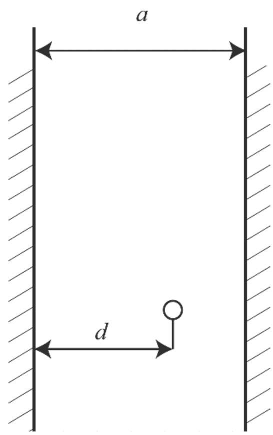

Fig. 3 – *Single thin line conductor between two ground planes.*

If $a \to +\infty$ in (15) then the capacitance of a wire residing at height *d* above a ground plane is

$$C' = \frac{2\pi\varepsilon_0}{\ln\dfrac{2d}{R}},$$

which can be readily confirmed by the method of images.

3.2 Capacitance of a two-wire line placed symmetrically in the groove in the horizontal plane

The geometry in this case is shown in Fig. 4.

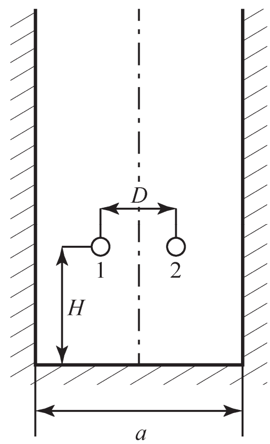

Fig. 4 – *Two-wire line inside the groove in the horizontal plane.*

Due to symmetry $V_2 = -V_1$ and the capacitance is

$$C' = \frac{1}{2V_1}\,. \tag{16}$$

The potential V_1 of conductor 1 is due to its own charge $q' = +1$ and to the charge $q' = -1$ of conductor 2

$$V_1 = V_1^{(1)} + V_1^{(2)}\,. \tag{17}$$

The potential $V_1^{(1)}$ is found from (13) if we put $d = \left(a - D\right)/2$

$$V_1^{(1)} = \frac{1}{2\pi\varepsilon_0}\ln\left(\frac{2a}{\pi R}\frac{\cos\dfrac{\pi D}{2a}\sinh\dfrac{\pi H}{a}}{\sqrt{\cos^2\dfrac{\pi D}{2a}+\sinh^2\dfrac{\pi H}{a}}}\right). \tag{18}$$

The potential $V_1^{(1)}$ of conductor 1 due the conductor 2 is the same (by reciprocity) as the potential of conductor 2 due to conductor 1 and can be found from (12) if we put $x = (a+D)/2$, $x' = (a-D)/2$, $y = y' = H$

$$\begin{aligned} V_1^{(2)} &= -\frac{1}{4\pi\varepsilon_0}\ln\left(\frac{2}{1-\cos\dfrac{\pi D}{a}}\cdot\frac{\cosh\dfrac{2\pi H}{a}-\cos\dfrac{\pi D}{a}}{\cosh\dfrac{2\pi H}{a}+1}\right) = \\ &= -\frac{1}{2\pi\varepsilon_0}\ln\frac{\sqrt{\sin^2\dfrac{\pi D}{2a}+\sinh^2\dfrac{\pi H}{a}}}{\sin\dfrac{\pi D}{2a}\cosh\dfrac{\pi H}{a}}. \end{aligned} \tag{19}$$

From (16) – (19) the capacitance is

$$C' = \frac{\pi\varepsilon_0}{\ln\left(\dfrac{a}{\pi R}\dfrac{\sin\dfrac{\pi D}{a}\sinh\dfrac{2\pi H}{a}}{\sqrt{\sin^2\dfrac{\pi D}{a}+\sinh^2\dfrac{2\pi H}{a}}}\right)}\,. \tag{20}$$

From (20) we can derive some special cases. They include the capacitance of a two-wire line spaced symmetrically between two ground planes ($H \to \infty$ in (20)) (Fig. 5), the capacitance of a two-wire line residing at height H over a ground plane ($a \to \infty$ in (20)), and the capacitance of a two-wire line in free space ($H \to \infty$ and $a \to \infty$ in (20)). These capacitances are:

$$C' = \frac{\pi\varepsilon_0}{\ln\left(\dfrac{a}{\pi R}\sin\dfrac{\pi D}{a}\right)},$$

$$C' = \frac{\pi\varepsilon_0}{\ln\dfrac{2DH}{R\sqrt{D^2+(2H)^2}}},$$

and

$$C' = \frac{\pi\varepsilon_0}{\ln\dfrac{D}{R}},$$

respectively. For these three special cases the image theory can also be used, but its implementation for the case shown in Fig. 5 is not straightforward, since the number of images is infinite.

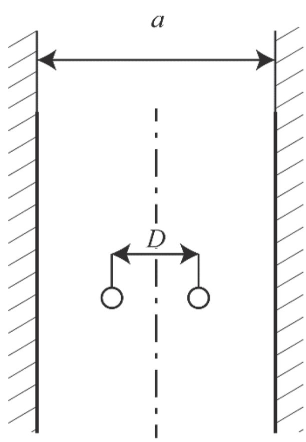

Fig. 5 – *Two-wire line spaced symmetrically between two ground planes.*

3.3 Capacitance of a two-wire line placed symmetrically in the groove in the vertical plane

The related geometry is shown in Fig. 6.

In this case $V_2 \neq -V_1$ and both potentials must be found separately

$$V_1 = V_1^{(1)} + V_1^{(2)},$$

$$V_2 = V_2^{(1)} + V_2^{(2)},$$

where $V_1^{(1)}, V_1^{(2)}, V_2^{(1)}$ and $V_2^{(2)}$ are found by properly using (12). We omit the details and present the final result

$$C' = \frac{1}{V_1 - V_2} = \frac{\pi\varepsilon_0}{\ln\left(\dfrac{2a}{\pi R}\dfrac{\tanh\dfrac{\pi D}{2a}}{\tanh\dfrac{\pi(D+2H)}{2a}}\sqrt{\tanh\dfrac{\pi H}{a}\tanh\dfrac{\pi(D+H)}{a}}\right)}. \quad (21)$$

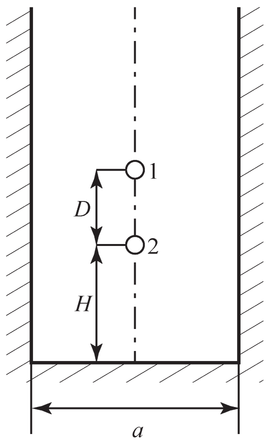

Fig. 6 – *Two-wire line inside the groove in the vertical plane.*

From (21) we obtain as special cases the capacitance of a two-wire line symmetrically spaced in the plane parallel to two ground planes ($H \to \infty$ in (21)) (Fig. 7)

$$C' = \frac{\pi\varepsilon_0}{\ln\left(\dfrac{2a}{\pi R}\tanh\dfrac{\pi D}{2a}\right)},$$

and the capacitance of a two-wire line spaced vertically above a ground plane ($a \to \infty$ in (21)), (Fig. 8)

$$C' = \frac{\pi\varepsilon_0}{\ln\dfrac{2D\sqrt{H(D+H)}}{R(D+2H)}}.$$

The latter result can also be obtained by using the method of images.

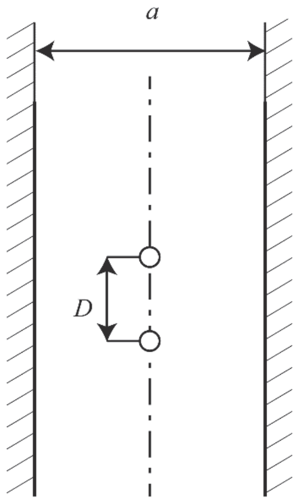

Fig. 7 – *Two-wire line spaced symmetrically in the plane parallel to two ground planes.*

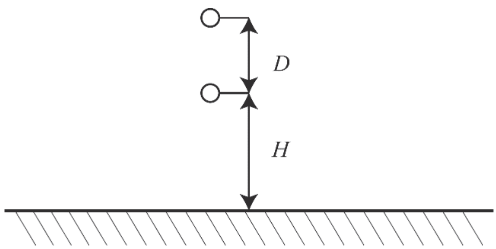

Fig. 8 – *Two-wire line spaced vertically above a ground plane.*

4 Conclusion

In this paper we derived Green's function for the semi-infinite strip in the form of an improper integral which is in fact an infinite summation of continuous harmonics in distinction to the standard form of Green's function comprising an infinite series of discrete harmonics. The obtained Green's function, after evaluating the integral in a closed form, is used to find formulas for the capacitances of a single wire line and some two-wire line configurations inside a grounded groove of infinite depth and length. Some of these formulas are checked against the available ones found in handbook literature.

5 Appendix

In this appendix we evaluate the integral in (8)

$$
\begin{aligned}
&I_1(A,B,C,D,F)=\int_0^\infty \frac{\sinh Ak \sinh Bk}{k \sinh Ck}\sin Dk \sin Fk \,\mathrm{d}k,\\
&A+B<C,\ A=a-x',\ B=x,\ C=a,\ D=y',\ F=y.
\end{aligned} \tag{A1}
$$

The starting point is the integral [9]

$$
I_2(A,C)=\int_0^\infty \frac{\sinh Ak}{\sinh Ck}\mathrm{d}k=\frac{\pi}{2C}\tan\frac{\pi A}{2C}, \quad A<C. \tag{A2}
$$

The next step is evaluation of

$$
I_3(A,B,C)=\int_0^\infty \frac{\sinh Ak \sinh Bk}{k \sinh Ck}\mathrm{d}k, \quad A+B<C. \tag{A3}
$$

Differentation of I_3 with respect to B gives

$$
\begin{aligned}
\frac{\partial I_3}{\partial B}&=\int_0^\infty \frac{\sinh Ak \cosh Bk}{\sinh Ck}\mathrm{d}k=\\
&=\frac{1}{2}\int_0^\infty \frac{\sinh\left(A+B\right)k+\sinh\left(A-B\right)k}{\sinh Ck}\mathrm{d}k=\\
&=\frac{\pi}{4C}\left(\tan\frac{\pi\left(A+B\right)}{2C}+\tan\frac{\pi\left(A-B\right)}{2C}\right),
\end{aligned} \tag{A4}
$$

where we used (A2). By integration, from (14) we find

$$
\begin{aligned}
I_3\left(A,B,C\right)&=\frac{\pi}{4C}\int\left(\tan\frac{\pi\left(A+B\right)}{2C}+\tan\frac{\pi\left(A-B\right)}{2C}\right)\mathrm{d}B+K=\\
&=\frac{1}{2}\left(\ln\cos\frac{\pi\left(A-B\right)}{2C}-\ln\cos\frac{\pi\left(A+B\right)}{2C}\right)+K,
\end{aligned} \tag{A5}
$$

where K may depend on A and C. From (A3) we see that I_3 vanishes for $B = 0$, and from (A5) we find that $K = 0$.

Whence

$$
I_3\left(A,B,C\right)=\frac{1}{2}\ln\frac{\cos\dfrac{\pi\left(A-B\right)}{2C}}{\cos\dfrac{\pi\left(A+B\right)}{2C}}. \tag{A6}
$$

$$I_4(A,B,C,D)=\int_0^\infty \frac{\sinh Ak \sinh Bk}{k\sinh Ck}\cos Dk\,\mathrm{d}k=$$
$$=\mathrm{Re}\int_0^\infty \frac{\sinh Ak\sinh\left(B+jD\right)k}{k\sinh Ck}\,\mathrm{d}k= \tag{A7}$$
$$=\mathrm{Re}\,I_3(A,B+jD,C)=\frac{1}{4}\ln\frac{\cosh\frac{\pi D}{C}+\cos\frac{\pi\left(A-B\right)}{C}}{\cosh\frac{\pi D}{C}+\cos\frac{\pi\left(A+B\right)}{C}},$$

where we used (A6). Finally, the integral I_1 (A, B, C, D, F), given by (A1) is evaluated by using the identity

$$\sin Dk\sin Fk=\frac{1}{2}\left[\cos\left(D-F\right)k-\cos\left(D+F\right)k\right],$$

and formula (A7), yielding (11).

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